SECOND-ORDER DIFFERENTIAL EQUATIONS IN THE LAGUERRE-HAHN CLASS

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Abstract. Laguerre-Hahn families on the real line are characterized in terms of second-order differential equations with matrix coefficients for vectors involving the orthogonal polynomials and their associated polynomials, as well as in terms of second-order differential equation for the functions of the second kind. Some characterizations of the classical families are derived.

1. Introduction

The study of real orthogonal polynomial sequences, $\{P_n\}$, that are solutions of differential equations

$$\sum_{j=0}^{N} A_j y^{(j)} = 0 \tag{1}$$

where A_j are polynomials (that may depend on n), is connected to measure perturbation theory and spectral theory of differential operators [6]. The minimal order of a differential equation (1) having orthogonal polynomial solutions is $N = 2$ or $N = 4$ [14]. For the case $N = 2$ in (1), with A_2, A_1 not depending on n and $A_0 = \lambda$, where λ is some spectral (eigenvalue) parameter depending on n,

$$A_2 y'' + A_1 y' + \lambda y = 0\,, \tag{2}$$

it is known the classification of sequences of orthogonal polynomial solutions: $\{P_n\}$ must be, up to a linear change of variable, a member of the classical families, that is, the Hermite, Laguerre, Jacobi and Bessel orthogonal polynomials (see [3] and also [12], for an overview on the problem of determination of orthogonal polynomial families that are solutions of (2)).

In the present paper we focus our attention on differential equations satisfied by Laguerre-Hahn orthogonal polynomials on the real line. These polynomials

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are related to Stieltjes functions satisfying Riccati type differential equations with polynomial coefficients [10, 16, 17, 18, 21]

$$AS' = BS^2 + CS + D\,. \tag{3}$$

Note that the Laguerre-Hahn orthogonal polynomials are a generalization of the semi-classical orthogonal polynomials, since the later ones are related to (3) with $B \equiv 0$, the classical families appearing if, in addiction, $\deg(A) \leq 2$ and $\deg(C) = 1$ [20]. Laguerre-Hahn orthogonal polynomials can be generated by performing a perturbation on the Stieltjes function of semi-classical orthogonal polynomials or by doing a modification on the three-term recurrence relation coefficients of semi-classical orthogonal polynomials [1, 4, 9, 21]. Thus, some well-known examples of Laguerre-Hahn polynomials include the associated polynomials of semi-classical orthogonal polynomials [1, 4, 7, 27], as well as the co-recursive, co-dilated and co-modified polynomials [4, 15].

Laguerre-Hahn families of orthogonal polynomials are solutions of differential equations (1), where the minimal order is $N = 4$ [2, 10, 11, 14, 19, 23], thus when no simplification occurs, Laguerre-Hahn orthogonal polynomials satisfy

$$A_4 P_n^{(4)} + A_3 P_n^{(3)} + A_2 P_n'' + A_1 P_n' + A_0 P_n = 0\,.$$

In this work we start by reinterpreting a result of [10], by showing an equivalence between (3) and differential-difference equations with matrix coefficients

$$A\Psi_n' = \mathcal{M}_n \Psi_n + \mathcal{N}_n \Psi_{n-1}\,, \quad \Psi_n = \begin{bmatrix} P_{n+1} & P_n^{(1)} \end{bmatrix}^T\,, \;\; n \geq 0\,,$$

with $\{P_n\}$ the sequence of monic orthogonal polynomials related to (3) and $\{P_n^{(1)}\}$ the sequence of associated polynomials of the first kind (cf. Theorem 1). We prove the equivalence between (3) and differential-difference equations for the sequence of functions of the second kind $\{q_n\}$ (cf. Section 2),

$$Aq_n' = (l_{n-1} + \frac{C}{2} + BS)q_n + \Theta_{n-1} q_{n-1}\,, \quad n \geq 0\,.$$

Next, we prove the equivalence between (3) and a second-order differential equation with matrix coefficients having polynomial entries,

$$\tilde{\mathcal{A}}_n \Psi_n'' + \tilde{\mathcal{B}}_n \Psi_n' + \tilde{\mathcal{C}}_n \Psi_n = 0_{2\times 1}\,, \quad n \geq 1\,, \tag{4}$$

as well as the equivalence between (3) and a second-order differential equation for the sequence of functions of the second kind $\{q_n\}$,

$$\tilde{A}_n q_n'' + \tilde{B}_n q_n' + \tilde{C}_n q_n = 0\,, \quad n \geq 1\,, \tag{5}$$

where $\tilde{A}_n$ is a polynomial and $\tilde{B}_n, \tilde{C}_n$ are functions. These equivalences are the analogue ones, for orthogonality on the real line, of [5, Theorems 1 and 2]. Taking into account the above referred equivalence between (3) and (4), we deduce a characterization of the sequences $\{\Psi_n\}$ corresponding to the Laguerre-Hahn class zero (i.e., $\max\{\deg(A), \deg(B)\} \leq 2$ and $\deg(C) = 1$ in (3) [4, 10]) as solutions of second-order matrix operators,

$$\mathbb{L}_n(\Psi_n) = 0\,, \quad \mathbb{L}_n = \mathcal{A}\,\mathbb{D}^2 + \Psi\,\mathbb{D} + \Lambda_n\,\mathbb{I}\,, \quad n \geq 0\,, \tag{6}$$

with $\mathcal{A}, \Psi, \Lambda_n$ 2×2 matrices explicitly given in terms of the polynomials A, B, C, D in (3) (cf. Theorem 3).

Finally, the last part of the paper is devoted to the analysis of the classical families. As a consequence of the above referred results some characterizations for the classical orthogonal polynomials are shown, from which we emphasize the characterizations in term of:

– the hypergeometric-type differential equation for the sequence of functions of the second kind;

– the differential equation that links the associated polynomial $P_n^{(1)}$ and the derivative of P_{n+1},

$$A\left(P_n^{(1)}\right)'' + (A' - C)\left(P_n^{(1)}\right)' + \lambda_{n+1}^* P_n^{(1)} = 2DP'_{n+1}\,, \quad n \geq 0\,,$$

where λ_{n+1}^* are constants, explicitly given in terms of A, B, C, D in (3).

This paper is organized as follows. In Section 2 we give the definitions and state the basic results which will be used in the forthcoming sections. In Section 3 we establish the equivalence between (3) and the second-order differential equations (4) and (5). In Section 4 we establish a characterization of Laguerre-Hahn orthogonal polynomials of class zero as solutions of (6). In Section **??** we present characterizations of the classical families of orthogonal polynomials.

2. Preliminary results

Let $\mathbb{P} = \text{span}\,\{z^k : k \in \mathbb{N}_0\}$ be the space of polynomials with complex coefficients and let $\mathbb{P}'$ be its algebraic dual space, i.e., the linear space of linear functionals defined on $\mathbb{P}$. We will denote by $\langle u, f\rangle$ the action of $u \in \mathbb{P}'$ on $f \in \mathbb{P}$. We consider a linear functional $u \in \mathbb{P}'$ and $\langle u, x^n\rangle = u_n,\ n \geq 0$, its *moments*. We will take u normalized, that is, $u_0 = 1$.

Given the sequence of moments (u_n) of u, the principal minors of the corresponding Hankel matrix are defined by $H_n = \det((u_{i+j})_{i,j=0}^n), n \geq 0$, where, by convention, $H_{-1} = 1$. The linear functional u is said to be *quasi-definite* (respectively, *positive-definite*) if $H_n \neq 0$ (respectively, $H_n > 0$), for all integer $n \geq 0$. If u is positive-definite, then it has an integral representation in terms of a positive Borel measure, μ, supported on an infinite set of points of the real line, I, such that

$$\langle u, x^n\rangle = \int_I x^n\, d\mu\,, \ \ n \geq 0\,.$$

Definition 1. Let $u \in \mathbb{P}'$. A sequence $\{P_n\}_{n\geq 0}$ is said to be orthogonal with respect to u if the following two conditions hold:
(i) $\deg(P_n) = n,\ n \geq 0\,,$
(ii) $\langle u, P_n P_m\rangle = k_n \delta_{n,m},\ k_n = \langle u, P_n^2\rangle \neq 0,\ n \geq 0\,.$
If the leading coefficient of each P_n is 1, then $\{P_n\}$ is said to be a sequence of monic orthogonal polynomials with respect to u, and it will be denoted by SMOP.

The equivalence between the quasi-definiteness of $u \in \mathbb{P}'$ and the existence of a SMOP with respect to u is well-known in the literature of orthogonal polynomials [8, 26].

Monic orthogonal polynomials satisfy a three-term recurrence relation [26]

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x),\ n = 1, 2, \ldots \tag{7}$$

with $P_0(x) = 1,\ P_1(x) = x - \beta_0$ and $\gamma_n \neq 0,\ n \geq 1,\ \gamma_0 = u_0 = 1$. Conversely, given a SMOP $\{P_n\}$ satisfying a three-term recurrence relation as above, there exists a unique quasi-definite linear functional u such that $\{P_n\}$ is the SMOP with respect to u [8, 26].

Definition 2. Let $\{P_n\}$ be the SMOP with respect to a linear functional u. The sequence of *associated polynomials of the first kind* is defined by

$$P_n^{(1)}(x) = \langle u_t, \frac{P_{n+1}(x) - P_{n+1}(t)}{x - t} \rangle \,, \quad n \geq 0 \,,$$

where u_t denotes the action of u on the variable t.

Note that the sequence $\{P_n^{(1)}\}$ also satisfies a three-term recurrence relation,

$$P_n^{(1)}(x) = (x - \beta_n) P_{n-1}^{(1)}(x) - \gamma_n P_{n-2}^{(1)}(x), \; n = 1, 2, \ldots$$

with $P_{-1}^{(1)}(x) = 0, \; P_0^{(1)}(x) = 1$.

Definition 3. Let $u \in \mathbb{P}'$ be quasi-definite and (u_n) its sequence of moments. The formal *Stieltjes function* of u is defined by

$$S(x) = \sum_{n=0}^{+\infty} \frac{u_n}{x^{n+1}} \,.$$

Given a SMOP $\{P_n\}$ and $\{P_n^{(1)}\}$ its sequence of associated polynomials, let S and $S^{(1)}$ denote the corresponding Stieltjes functions, respectively. One has

$$\gamma_1 S^{(1)}(x) = -\frac{1}{S(x)} + (x - \beta_0) \,. \tag{8}$$

The sequence of *functions of the second kind* corresponding to $\{P_n\}$ is defined as follows:

$$q_{n+1}(x) = P_{n+1}(x) S(x) - P_n^{(1)}(x) \,, \quad n \geq 0 \,, \; q_0 = S \,. \tag{9}$$

In the positive-definite case, if u is defined in terms of a measure μ, one has [25]

$$q_n(x) = \int \frac{P_n(t)}{x - t} d\mu(t) \,, \; x \notin supp(\mu) \,, \quad n \geq 0 \,.$$

Definition 4. Let $u \in \mathbb{P}'$ be quasi-definite and let S be its Stieltjes function. u (or S) is said to be Laguerre-Hahn if there exist polynomials A, B, C, D, with $A \neq 0$, such that S satisfies a Riccati differential equation

$$AS' = BS^2 + CS + D. \tag{10}$$

The corresponding sequence of orthogonal polynomials is called Laguerre-Hahn. If $B = 0$, then S is said to be *semi-classical* or *affine Laguerre-Hahn*.

Note that if u is semi-classical, with the corresponding Stieltjes function satisfying $AS' = CS + D$, then, taking into account (8), there follows that $S^{(1)}$ is Laguerre-Hahn, since it satisfies a Riccati type differential equation.

Equation (10) is equivalent to the distributional equation for the corresponding linear functional u

$$\mathcal{D}(Au) = \psi u + B(x^{-1} u^2) \,, \tag{11}$$

where $\psi = A' + C$ [21].

The distributional equation (11) is not unique, many triples of polynomials can be associated with such an equation, but only one canonical set of minimal degree exists. The *class* of u is defined as the minimum value of $\max\{\deg(\psi)-1, d-2\}, \; d = \max\{\deg(A), \deg(B)\}$, for all triples of polynomials satisfying (11). When $B \equiv 0$ and the class of u is zero, that is, $\deg(\psi) = 1$ and $\deg(A) \leq 2$, u is called a *classical functional*, and the corresponding orthogonal polynomials are the classical orthogonal polynomials.

In the sequel we will use the following matrices:

$$\Psi_n = \begin{bmatrix} P_{n+1} \\ P_n^{(1)} \end{bmatrix}, \quad \mathcal{Q}_n = \begin{bmatrix} q_{n+1} \\ q_n \end{bmatrix}, \quad n \geq 0 \,. \tag{12}$$

Hereafter I denotes the 2×2 identity matrix.

Lemma 1. *Let $u \in \mathbb{P}'$ be quasi-definite, let $\{P_n\}$ be the corresponding SMOP and β_n, γ_n the coefficients of the recurrence relation (7). Let $\{\Psi_n\}$, $\{\mathcal{Q}_n\}$ be the sequences defined in (12). Then,*
(a) Ψ_n satisfies

$$\Psi_n = (x - \beta_n)\Psi_{n-1} - \gamma_n \Psi_{n-2} \,, \quad n \geq 1 \,, \tag{13}$$

with initial conditions $\Psi_{-1} = \begin{bmatrix} P_0 \\ P_{-1}^{(1)} \end{bmatrix}, \Psi_0 = \begin{bmatrix} P_1 \\ P_0^{(1)} \end{bmatrix}$;

(b) $\varphi_n = \begin{bmatrix} \Psi_{n+1} \\ \Psi_n \end{bmatrix}$ *satisfies*

$$\varphi_n = \mathcal{K}_n \varphi_{n-1} \,, \quad \mathcal{K}_n = \begin{bmatrix} (x - \beta_{n+1})I & -\gamma_{n+1} I \\ I & 0_{2\times 2} \end{bmatrix}, \quad n \geq 1 \,, \tag{14}$$

with initial conditions $\varphi_0 = \begin{bmatrix} P_2 & P_1^{(1)} & P_1 & P_0^{(1)} \end{bmatrix}^T$;
(c) $\mathcal{Q}_n$ satisfies

$$\mathcal{Q}_n = \mathcal{A}_n \mathcal{Q}_{n-1} \,, \quad n \geq 1 \,, \tag{15}$$

with $\mathcal{A}_n = \begin{bmatrix} x - \beta_n & -\gamma_n \\ 1 & 0 \end{bmatrix}$ *and initial conditions* $\mathcal{Q}_0 = \begin{bmatrix} (x - \beta_0)S - 1 \\ S \end{bmatrix}$.

Throughout the text, $X^{(i,j)}$ will denote the (i, j) entry in the matrix X.

3. Second-order differential equations with matrix coefficients

Theorem 1. *Let $u \in \mathbb{P}'$ be quasi-definite and let S be its Stieltjes function. Let $\{\Psi_n\}$ be the corresponding sequence defined in (12), and let $\{q_n\}$ be the sequence of functions of the second kind. The following statements are equivalent:*
(a) S satisfies

$$AS' = BS^2 + CS + D \,, \quad A, B, C, D \in \mathbb{P} \,;$$

(b) Ψ_n satisfies

$$A\Psi_n' = \mathcal{M}_n \Psi_n + \mathcal{N}_n \Psi_{n-1} \,, \quad n \geq 0 \,, \tag{16}$$

where $\mathcal{M}_n = \begin{bmatrix} l_n - \frac{C}{2} & -B \\ D & l_n + \frac{C}{2} \end{bmatrix}$, $\mathcal{N}_n = \Theta_n I$, *and Θ_n, l_n are bounded degree polynomials, with initial conditions*

$$A = (l_0 - C/2)(x - \beta_0) - B + \Theta_0 \,, \; 0 = (x - \beta_0)D + (l_0 + C/2) \,;$$

(c) q_n satisfies

$$Aq_n' = (l_{n-1} + \frac{C}{2} + BS)q_n + \Theta_{n-1} q_{n-1} \,, \quad n \geq 0 \,, \tag{17}$$

with $q_{-1}=1,\ \Theta_{-1}=D,\ l_{-1}=C/2$.

Moreover, the polynomials l_n *and* Θ_n *involved in Eqs. (16) and (17) satisfy the following relations, for all* $n\geq 0$*:*

$$l_{n+1}+l_n=-\frac{(x-\beta_{n+1})}{\gamma_{n+1}}\Theta_n\,, \tag{18}$$

$$\Theta_{n+1}=A+\frac{\gamma_{n+1}}{\gamma_n}\Theta_{n-1}+(x-\beta_{n+1})(l_n-l_{n+1})\,. \tag{19}$$

Proof. $(a)\Leftrightarrow(b)$.
This equivalence was proven in [10]. Indeed, $(a)\Rightarrow(b)$ is obtained by using the following facts:
- Theorem 3.1, page 64, showing the equivalence between the Riccati equation for the formal Stieltjes function and functional equation;
- the equivalence between the Riccati Equation for the formal Stieltjes function and Eq. (1.11), page 75 (Theorem 1.1), in our notation,

$$AP'_{n+1}=-BP_n^{(1)}+\sum_{\mu=n-e+1}^{n+l}\theta_{n,\mu}P_\mu\,,\quad n\geq e\,, \tag{20}$$

with $l=\max\{t,b\},e=\max\{s+1,b-1\},t=\deg(A),b=\deg(B),s=\max\{p,q-1\}-1,p=\deg(A'+C),q=\deg(x(A'+C)-A)$;
- the fact that the former equivalence leads to Eq. (2.12), page 82, in our notation,

$$A\left(P_{n+1}^{(1)}\right)'=DP_{n+2}-(x-\beta_0)DP_{n+1}^{(1)}+\sum_{\mu=n-\tilde{e}+1}^{n+\tilde{l}}\tilde{\theta}_{n,\mu}P_\mu\,,\quad n\geq\tilde{e}\,. \tag{21}$$

- the fact that Eqs. (20) and (21) can be written, using the three term recurrence relation, as Eqs. (3.9) and (3.10), page 83, in our notation,

$$\begin{aligned} AP'_{n+1}&=(l_n-C/2)P_{n+1}-BP_n^{(1)}+\Theta_nP_n\,,\quad n\geq 0\,,\\ A\left(P_n^{(1)}\right)'&=DP_{n+1}+(l_n+C/2)P_n^{(1)}+\Theta_nP_{n-1}^{(1)}\,,\quad n\geq 0\,,\end{aligned} \tag{22}$$

which is (16).

The proof of $(b)\Rightarrow(a)$ follows by applying successively the three term recurrence relation in (22), rearranging as (20).

$(b)\Rightarrow(c)$.
Take derivatives in (9), then multiply the resulting expression by A and use the equations enclosed in (16), as well as the Riccati equation for S, to get

$$Aq'_n=(l_{n-1}+\frac{C}{2}+BS)q_n+\Theta_{n-1}q_{n-1}\,,\ n\geq 1\,.$$

Furthermore, as $q_{-1}=1,\ q_0=S$, taking into account $\Theta_{-1}=D,\ l_{-1}=C/2$, there follows that the above equation also holds for $n=0$. Hence we obtain (17).
$(c)\Rightarrow(a)$.
Take $n=0$ in (17), with $q_{-1}=1,\ q_0=S,\ \Theta_{-1}=D,\ l_{-1}=C/2$.

Let us now deduce the relations (18) and (19) for the polynomials l_n,Θ_n involved in Eqs. (16) and (17).

We write (16) for $n+1$ and use the recurrence relation (13), thus getting

$$A\Psi_n + (x-\beta_{n+1})A\Psi'_n - \gamma_{n+1}A\Psi'_{n-1} \\ = (x-\beta_{n+1})\mathcal{M}_{n+1}\Psi_n - \gamma_{n+1}\mathcal{M}_{n+1}\Psi_{n-1} + \mathcal{N}_{n+1}\Psi_n\,.$$

The use of (16) for n as well as for $n-1$ in the above equation yields

$$A_n\Psi_n = B_n\Psi_{n-1}\,,\quad n\geq 0\,. \tag{23}$$

where A_n and B_n are the polynomials given by

$$A_n = A + \frac{\gamma_{n+1}}{\gamma_n}\Theta_{n-1} + (x-\beta_{n+1})(l_n - l_{n+1}) - \Theta_{n+1}\,,$$
$$B_n = -(x-\beta_{n+1})\Theta_n + \gamma_{n+1}(l_{n-1} - l_{n+1}) + \frac{\gamma_{n+1}}{\gamma_n}(x-\beta_n)\Theta_{n-1}\,.$$

Note that (23) reads as

$$A_nP_{n+1} = B_nP_n\,,\ \ A_nP_n^{(1)} = B_nP_{n-1}^{(1)}\,,\quad n\geq 0\,. \tag{24}$$

By multiplying the first equation in (24) by $P_{n-1}^{(1)}$ and the second one by P_n and by subtracting the corresponding result, we get

$$A_n(P_n^{(1)}P_n - P_{n+1}P_{n-1}^{(1)}) = 0\,,\quad n\geq 0\,.$$

Also, by multiplying the first equation in (24) by $P_n^{(1)}$ and the second one by P_{n+1} and by subtracting the corresponding result, we get

$$B_n(P_n^{(1)}P_n - P_{n+1}P_{n-1}^{(1)}) = 0\,,\quad n\geq 0\,.$$

As $P_n^{(1)}P_n - P_{n+1}P_{n-1}^{(1)} = \prod_{k=0}^n \gamma_k \neq 0$, $n\geq 0$ [26], there follows $A_n = B_n = 0$, $n\geq 0$, that is, we have (19) as well as

$$l_{n+1} + \frac{(x-\beta_{n+1})}{\gamma_{n+1}}\Theta_n = l_{n-1} + \frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1}\,.$$

Let us add l_n in both hand sides of the above equation. We obtain

$$m_n = m_{n-1}\,,\quad m_n = l_{n+1} + l_n + \frac{(x-\beta_{n+1})}{\gamma_{n+1}}\Theta_n\,,\quad n\geq 0\,,$$

from which there follows $m_n = m_{-1}$, for all $n\geq 0$, that is,

$$l_{n+1} + l_n + \frac{(x-\beta_{n+1})}{\gamma_{n+1}}\Theta_n = l_0 + l_{-1} + \frac{(x-\beta_0)}{\gamma_0}\Theta_{-1}\,,\quad n\geq 0\,.$$

Using the initial conditions $D=\Theta_{-1}, C/2 = l_{-1}$ we get $l_0 + l_{-1} + \frac{(x-\beta_0)}{\gamma_0}\Theta_{-1} = 0$, thus (18) follows. □

If we take $B=0$ in the previous theorem we obtain differential relations in the semi-classical class.

Corollary 1. *Let $u\in\mathbb{P}'$ be quasi-definite and let S be its Stieltjes function. Let $\{\Psi_n\}$ be the corresponding sequence defined in (12), and let $\{q_n\}$ be the sequence of functions of the second kind. The following statements are equivalent:*
(a) S is semi-classical and it satisfies $AS' = CS + D$;
(b) Ψ_n satisfies

$$A\Psi'_n = \mathcal{M}_n\Psi_n + \mathcal{N}_n\Psi_{n-1}\,,\quad n\geq 0\,,$$

where $\mathcal{M}_n = \begin{bmatrix} l_n - \frac{C}{2} & 0 \\ D & l_n + \frac{C}{2} \end{bmatrix}$, $\mathcal{N}_n = \Theta_n I$, *and* Θ_n, l_n *are bounded degree polynomials;*
(c) q_n *satisfies the differential-difference equation with polynomial coefficients*

$$A q_n' = (l_{n-1} + \frac{C}{2}) q_n + \Theta_{n-1} q_{n-1}\,, \quad n \geq 0\,.$$

Theorem 2. *Let* $u \in \mathbb{P}'$ *be quasi-definite and let* S *be its Stieltjes function. Let* $\{\Psi_n\}$ *be the corresponding sequence defined in (12), and let* $\{q_n\}$ *be the sequence of functions of the second kind. The following statements are equivalent:*
(a) S *satisfies*

$$AS' = BS^2 + CS + D\,, \quad A, B, C, D \in \mathbb{P}\,;$$

(b) Ψ_n *satisfies the second-order differential equation*

$$\tilde{\mathcal{A}}_n \Psi_n'' + \tilde{\mathcal{B}}_n \Psi_n' + \tilde{\mathcal{C}}_n \Psi_n = 0_{2\times 1}\,, \quad n \geq 1\,, \tag{25}$$

where $\tilde{\mathcal{A}}_n, \tilde{\mathcal{B}}_n, \tilde{\mathcal{C}}_n$ *are matrices, with polynomial entries, given by*

$$\tilde{\mathcal{A}}_n = A^2 \Theta_n I\,, \tag{26}$$

$$\tilde{\mathcal{B}}_n = A\Theta_n (A' I - \mathcal{M}_n - \mathcal{M}_{n-1}) - A\Theta_{n-1}\Theta_n \frac{(x-\beta_n)}{\gamma_n} I - A^2 \Theta_n' I\,, \tag{27}$$

$$\begin{aligned} \tilde{\mathcal{C}}_n = {} & \Theta_n \left(\frac{\Theta_{n-1}\Theta_n}{\gamma_n} I - A\mathcal{M}_n' \right) \\ & + \left\{ \Theta_n \left(\mathcal{M}_{n-1} + \frac{(x-\beta_n)}{\gamma_n} \Theta_{n-1} I \right) + A\Theta_n' I \right\} \mathcal{M}_n\,; \end{aligned} \tag{28}$$

(c) q_n *satisfies the second-order differential equation*

$$\tilde{A}_n q_{n+1}'' + \tilde{B}_n q_{n+1}' + \tilde{C}_n q_{n+1} = 0\,, \quad n \geq 0\,, \tag{29}$$

where $\tilde{A}_n$, $\tilde{B}_n, \tilde{C}_n$ *are functions given by*

$$\tilde{A}_n = A^2 \Theta_n\,, \tag{30}$$

$$\tilde{B}_n = A\Theta_n (A' - C - 2BS) - A^2 \Theta_n'\,, \tag{31}$$

$$\begin{aligned} \tilde{C}_n = {} & \Theta_n \left(\frac{\Theta_{n-1}\Theta_n}{\gamma_n} - A(l_n + \frac{C}{2} + BS)' \right) \\ & + (l_n + \frac{C}{2} + BS) \left(\Theta_n(-l_n + \frac{C}{2} + BS) + A\Theta_n' \right)\,. \end{aligned} \tag{32}$$

Corollary 2. *Let* $u \in \mathbb{P}'$ *be quasi-definite and let* S *be its Stieltjes function. Let* $\{\Psi_n\}$ *be the corresponding sequence defined in (12), and let* $\{q_n\}$ *be the sequence of functions of the second kind. The following statements are equivalent:*
(a) S *is semi-classical and satisfies*

$$AS' = CS + D\,, \quad A, C, D \in \mathbb{P}\,;$$

(b) $\{\Psi_n\}$ satisfies the second-order differential equation (25) with matrix coefficients of polynomial entries given by

$$\tilde{\mathcal{A}}_n = A^2\Theta_n I \,, \tag{33}$$

$$\tilde{\mathcal{B}}_n = A\begin{bmatrix}(A'+C)\Theta_n - A\Theta_n' & 0 \\ -2D\Theta_n & (A'-C)\Theta_n - A\Theta_n'\end{bmatrix} , \tag{34}$$

$$\tilde{\mathcal{C}}_n = \begin{bmatrix} g_n + (l_n - C/2)h_n & 0 \\ A(D\Theta_n' - D'\Theta_n) & (g_n - AC'\Theta_n) + (l_n + C/2)(h_n + C\Theta_n)\end{bmatrix} , \tag{35}$$

where

$$g_n = \Theta_n\left(\frac{\Theta_{n-1}\Theta_n}{\gamma_n} - A(l_n - C/2)'\right) ,$$

$$h_n = \Theta_n\left(l_{n-1} - C/2 + \frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1}\right) + A\Theta_n' ;$$

(c) $\{q_n\}$ satisfies the second-order differential equation (29) with polynomial coefficients $\tilde{A}_n, \tilde{B}_n, \tilde{C}_n$ given by

$$\tilde{A}_n = A\Theta_n \,, \tag{36}$$

$$\tilde{B}_n = \Theta_n(A' - C) - A\Theta_n' \,, \tag{37}$$

$$\tilde{C}_n = \Theta_n\left(\sum_{k=1}^{n}\frac{\Theta_{k-1}}{\gamma_k} + D - (l_n + \frac{C}{2})'\right) + \Theta_n'(l_n + \frac{C}{2}) \,. \tag{38}$$

Proof. Taking $B \equiv 0$ in the previous theorem and using the relations (18), the matrices $\tilde{\mathcal{A}}_n$-$\tilde{\mathcal{C}}_n$ given by (26)-(28) simplify as (33)-(35).

Let us now deduce the coefficients (36)-(38). Taking $B \equiv 0$ in the previous theorem, there follows the second-order differential equation (29) with polynomial coefficients $\tilde{A}_n$-$\tilde{C}_n$ given by

$$\begin{aligned}
\tilde{A}_n &= A^2\Theta_n \,, \\
\tilde{B}_n &= A\left(\Theta_n(A'-C) - A\Theta_n'\right) , \\
\tilde{C}_n &= \Theta_n\left(\frac{\Theta_{n-1}\Theta_n}{\gamma_n} - l_n^2 + (C/2)^2\right) - A\left(\Theta_n(l_n + C/2)' - \Theta_n'(l_n + C/2)\right) .
\end{aligned}$$

Let $\tau_n = \dfrac{\Theta_{n-1}\Theta_n}{\gamma_n} - l_n^2 + (C/2)^2$.

Using (18) and (19) we obtain

$$\frac{\Theta_{n-1}\Theta_n}{\gamma_n} - l_n^2 = A\frac{\Theta_{n-1}}{\gamma_n} + \frac{\Theta_{n-2}\Theta_{n-1}}{\gamma_{n-1}} - l_{n-1}^2 \,, \quad n \geq 1 \,,$$

thus,

$$\frac{\Theta_{n-1}\Theta_n}{\gamma_n} - l_n^2 = A\sum_{k=1}^{n}\frac{\Theta_{k-1}}{\gamma_k} + \frac{\Theta_{-1}\Theta_0}{\gamma_0} - l_0^2 \,, \quad n \geq 1 \,. \tag{39}$$

The initial conditions

$$\Theta_{-1} = D, \;\; (x-\beta_0)D + (l_0 + C/2) = 0, \;\; A = (l_0 - C/2)(x - \beta_0) - B + \Theta_0$$

yield

$$\frac{\Theta_{-1}\Theta_0}{\gamma_0} - l_0^2 = AD + BD - (C/2)^2 \,. \tag{40}$$

From (39) and (40) there follows

$$\tau_n = A\sum_{k=1}^{n}\frac{\Theta_{k-1}}{\gamma_k} + AD + BD\,. \tag{41}$$

Note that we are assuming $B\equiv 0$, thus we obtain $\tilde{C}_n$ given by

$$\tilde{C}_n = A\left\{\Theta_n\left(\sum_{k=1}^{n}\frac{\Theta_{k-1}}{\gamma_k} + D - (l_n+\frac{C}{2})'\right) + \Theta_n'(l_n+\frac{C}{2})\right\}.$$

Hence, the coefficients of the second-order differential equation (29) when $B\equiv 0$ are (36)-(38). □

We recover the well-known result that follows [10, 11].

Corollary 3. *Let S be a Laguerre-Hahn Stieltjes function satisfying $AS' = BS^2 + CS + D$. The SMOP related to S, $\{P_n\}$, as well as the sequence of associated polynomials of the first kind, $\{P_n^{(1)}\}$, satisfy fourth order linear differential equations with polynomial coefficients.*

Proof. Let us consider the differential equation (25) as well as its first and second derivatives,

$$\tilde{\mathcal{A}}_n\Psi_n^{(4)} + \mathcal{E}_n\Psi_n''' + \mathcal{F}_n\Psi_n'' + \mathcal{G}_n\Psi_n' + \mathcal{H}_n\Psi_n = 0_{2\times 1}\,, \tag{42}$$

$$\tilde{\mathcal{A}}_n\Psi_n''' + \mathcal{I}_n\Psi_n'' + \mathcal{J}_n\Psi_n' + \mathcal{K}_n\Psi_n = 0_{2\times 1}\,, \tag{43}$$

$$\tilde{\mathcal{A}}_n\Psi_n'' + \tilde{\mathcal{B}}_n\Psi_n' + \tilde{\mathcal{C}}_n\Psi_n = 0_{2\times 1}\,, \tag{44}$$

with

$$\mathcal{E}_n = 2\tilde{\mathcal{A}}_n' + \tilde{\mathcal{B}}_n\,,\ \mathcal{F}_n = \tilde{\mathcal{A}}_n'' + 2\tilde{\mathcal{B}}_n' + \tilde{\mathcal{C}}_n\,,\ \mathcal{G}_n = \tilde{\mathcal{B}}_n'' + 2\tilde{\mathcal{C}}_n'\,,\ \mathcal{H}_n = \tilde{\mathcal{C}}_n''\,,$$
$$\mathcal{I}_n = \tilde{\mathcal{A}}_n' + \tilde{\mathcal{B}}_n\,,\ \mathcal{J}_n = \tilde{\mathcal{B}}_n' + \tilde{\mathcal{C}}_n\,,\ \mathcal{K}_n = \tilde{\mathcal{C}}_n'\,.$$

The Eqs. (42), (43) and (44) enclose the following ones:

$$\mathbb{L}_{n,4}(P_{n+1}) = -\mathcal{E}_n^{(1,2)}(P_n^{(1)})''' - \mathcal{F}_n^{(1,2)}(P_n^{(1)})'' - \mathcal{G}_n^{(1,2)}(P_n^{(1)})' - \mathcal{H}_n^{(1,2)}P_n^{(1)}\,, \tag{45}$$

$$\mathbb{L}_{n,4}^{(1)}(P_n^{(1)}) = -\mathcal{E}_n^{(2,1)}(P_{n+1})''' - \mathcal{F}_n^{(2,1)}P_{n+1}'' - \mathcal{G}_n^{(2,1)}P_{n+1}' - \mathcal{H}_n^{(2,1)}P_{n+1}\,, \tag{46}$$

$$\mathbb{L}_{n,3}(P_{n+1}) = -\mathcal{I}_n^{(1,2)}(P_n^{(1)})'' - \mathcal{J}_n^{(1,2)}(P_n^{(1)})' - \mathcal{K}_n^{(1,2)}P_n^{(1)}\,, \tag{47}$$

$$\mathbb{L}_{n,3}^{(1)}(P_n^{(1)}) = -\mathcal{I}_n^{(2,1)}P_{n+1}'' - \mathcal{J}_n^{(2,1)}P_{n+1}' - \mathcal{K}_n^{(2,1)}P_{n+1}\,, \tag{48}$$

$$\mathbb{L}_{n,2}(P_{n+1}) = -\tilde{\mathcal{B}}_n^{(1,2)}(P_n^{(1)})' - \tilde{\mathcal{C}}_n^{(1,2)}P_n^{(1)}\,, \tag{49}$$

$$\mathbb{L}_{n,2}^{(1)}(P_n^{(1)}) = -\tilde{\mathcal{B}}_n^{(2,1)}P_{n+1}' - \tilde{\mathcal{C}}_n^{(2,1)}P_{n+1}\,, \tag{50}$$

with

$$\mathbb{L}_{n,4} = A^2\Theta_n\mathbb{D}^4 + \mathcal{E}_n^{(1,1)}\mathbb{D}^3 + \mathcal{F}_n^{(1,1)}\mathbb{D}^2 + \mathcal{G}_n^{(1,1)}\mathbb{D} + \mathcal{H}_n^{(1,1)}\mathbb{I}\,,$$
$$\mathbb{L}_{n,4}^{(1)} = A^2\Theta_n\mathbb{D}^4 + \mathcal{E}_n^{(2,2)}\mathbb{D}^3 + \mathcal{F}_n^{(2,2)}\mathbb{D}^2 + \mathcal{G}_n^{(2,2)}\mathbb{D} + \mathcal{H}_n^{(2,2)}\mathbb{I}\,,$$
$$\mathbb{L}_{n,3} = A^2\Theta_n\mathbb{D}^3 + \mathcal{I}_n^{(1,1)}\mathbb{D}^2 + \mathcal{J}_n^{(1,1)}\mathbb{D} + \mathcal{K}_n^{(1,1)}\mathbb{I}\,,$$
$$\mathbb{L}_{n,3}^{(1)} = A^2\Theta_n\mathbb{D}^3 + \mathcal{I}_n^{(2,2)}\mathbb{D}^2 + \mathcal{J}_n^{(2,2)}\mathbb{D} + \mathcal{K}_n^{(2,2)}\mathbb{I}\,,$$
$$\mathbb{L}_{n,2} = A^2\Theta_n\mathbb{D}^2 + \tilde{\mathcal{B}}_n^{(1,1)}\mathbb{D} + \tilde{\mathcal{C}}_n^{(1,1)}\mathbb{I}\,,$$
$$\mathbb{L}_{n,2}^{(1)} = A^2\Theta_n\mathbb{D}^2 + \tilde{\mathcal{B}}_n^{(2,2)}\mathbb{D} + \tilde{\mathcal{C}}_n^{(2,2)}\mathbb{I}\,.$$

By multiplying (45) by $A^2\Theta_n$ and using (48), and, in turn, by multiplying the resulting equation by $A^2\Theta_n$ and using (50), we obtain

$$\widehat{\widehat{\mathbb{L}}}_{n,4}(P_{n+1}) = \widehat{\widehat{U}}_n(P_n^{(1)})' + \widehat{\widehat{V}}_n P_n^{(1)}\,, \tag{51}$$

with

$$\begin{aligned}
&\widehat{\widehat{\mathbb{L}}}_{n,4} = A^2\Theta_n\widehat{\mathbb{L}}_{n,4} + R_n\tilde{\mathcal{B}}_n^{(2,1)}\mathbb{D} + R_n\tilde{\mathcal{C}}_n^{(2,1)}\mathbb{I}\,,\\
&\widehat{\widehat{U}}_n = -R_n\tilde{\mathcal{B}}_n^{(2,2)} + S_nA^2\Theta_n\,, \widehat{\widehat{V}}_n = -R_n\tilde{\mathcal{C}}_n^{(2,2)} + T_nA^2\Theta_n\,,\\
&\widehat{\mathbb{L}}_{n,4} = A^2\Theta_n\mathbb{L}_{n,4} - \mathcal{E}_n^{(1,2)}\mathcal{I}_n^{(2,1)}\mathbb{D}^2 - \mathcal{E}_n^{(1,2)}\mathcal{J}_n^{(2,1)}\mathbb{D} - \mathcal{E}_n^{(1,2)}\mathcal{K}_n^{(2,1)}\mathbb{I}\,,\\
&R_n = \mathcal{E}_n^{(1,2)}\mathcal{I}_n^{(2,2)} - A^2\Theta_n\mathcal{F}_n^{(1,2)}\,,\\
&S_n = \mathcal{E}_n^{(1,2)}\mathcal{J}_n^{(2,2)} - A^2\Theta_n\mathcal{G}_n^{(1,2)}\,,\\
&T_n = \mathcal{E}_n^{(1,2)}\mathcal{K}_n^{(2,2)} - A^2\Theta_n\mathcal{H}_n^{(1,2)}\,.
\end{aligned}$$

By multiplying (47) by $A^2\Theta_n$ and using (50), we obtain

$$\widehat{\mathbb{L}}_{n,3}(P_{n+1}) = \widehat{U}_n(P_n^{(1)})' + \widehat{V}_nP_n^{(1)}\,, \tag{52}$$

with

$$\begin{aligned}
&\widehat{\mathbb{L}}_{n,3} = A^2\Theta_n\mathbb{L}_{n,3} + \mathcal{I}_n^{(1,2)}\tilde{\mathcal{B}}_n^{(2,1)}\mathbb{D} + \mathcal{I}_n^{(1,2)}\tilde{\mathcal{C}}_n^{(2,1)}\mathbb{I}\,,\\
&\widehat{U}_n = \mathcal{I}_n^{(1,2)}\tilde{\mathcal{B}}_n^{(2,2)} - \mathcal{J}_n^{(1,2)}A^2\Theta_n\,, \widehat{V}_n = \mathcal{I}_n^{(1,2)}\tilde{\mathcal{C}}_n^{(2,2)} - \mathcal{K}_n^{(1,2)}A^2\Theta_n\,.
\end{aligned}$$

Eqs. (49), (51) and (52) yield the determinant that gives us the fourth-order differential equation for $\{P_n\}$

$$\begin{vmatrix}
\widehat{\widehat{\mathbb{L}}}_{n,4}(P_{n+1}) & \widehat{\widehat{U}}_n & \widehat{\widehat{V}}_n\\
\widehat{\mathbb{L}}_{n,3}(P_{n+1}) & \widehat{U}_n & \widehat{V}_n\\
\mathbb{L}_{n,2}(P_{n+1}) & -\tilde{\mathcal{B}}_n^{(1,2)} & -\tilde{\mathcal{C}}_n^{(1,2)}
\end{vmatrix} = 0\,, \quad n\geq 1\,.$$

The fourth-order differential equation for $\{P_n^{(1)}\}$ is deduced analogously. □

The lemmas that follow will be used to prove Theorem 2. The detailed proof of Theorem 2 will be given at the end of the section.

Lemma 2. *Let $u\in\mathbb{P}'$ be quasi-definite and let S be its Stieltjes function. Let $\{\Psi_n\}$ be the corresponding sequence defined in (12), and let $\{q_n\}$ be the sequence of functions of the second kind. If S satisfies $AS' = BS^2+CS+D\,,\ \ A,B,C,D\in\mathbb{P}\,,$ then $\{\Psi_n\}$ satisfies (25) with coefficients (26)-(28) and $\{q_n\}$ satisfies (29) with coefficients (30)-(32).*

Proof. If we take derivatives in (16) and multiply the resulting equation by A we get

$$A^2\Psi_n'' = A(\mathcal{M}_n - A'I)\Psi_n' + \mathcal{N}_nA\Psi_{n-1}' + A\mathcal{M}_n'\Psi_n + A\mathcal{N}_n'\Psi_{n-1}\,. \tag{53}$$

If we use (16) to $n-1$ and the recurrence relation (13) for Ψ_n we obtain

$$A\Psi_{n-1}' = \left(\mathcal{M}_{n-1} + \frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1}\right)\Psi_{n-1} - \frac{\Theta_{n-1}}{\gamma_n}\Psi_n\,. \tag{54}$$

The substitution of (54) into (53) yields

$$A^2\Psi_n'' = A(\mathcal{M}_n - A'I)\Psi_n' + \left(A\mathcal{M}_n' - \frac{\Theta_{n-1}}{\gamma_n}\mathcal{N}_n\right)\Psi_n + \left[\mathcal{N}_n\left(\mathcal{M}_{n-1} + \frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1}\right) + A\mathcal{N}_n'\right]\Psi_{n-1}\,.$$

The multiplication of the above equation by Θ_n and the use of (16) gives us (25) with coefficients (26)-(28).

To get (29) we proceed analogously as before, starting by taking derivatives in (17), thus obtaining $\tilde{A}_n q_{n+1}'' + \tilde{B}_n q_{n+1}' + \tilde{C}_n q_{n+1} = 0$, with $\tilde{A}_n = A^2\Theta_n$ and

$$\begin{aligned}
\tilde{B}_n &= -A\Theta_n(l_n + l_{n-1} + C + 2BS - A') - A\frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1}\Theta_n - A^2\Theta_n'\,,\\
\tilde{C}_n &= \Theta_n\left(\frac{\Theta_{n-1}\Theta_n}{\gamma_n} - A(l_n + \frac{C}{2} + BS)'\right)\\
&+ (l_n + \frac{C}{2} + BS)\left[\Theta_n\left(\frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1} + l_{n-1} + \frac{C}{2} + BS\right) + A\Theta_n'\right].
\end{aligned}$$

The use of $l_n + l_{n-1} = -\frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1}$ (cf. (18)) in the above equations yields $\tilde{B}_n$ and $\tilde{C}_n$ given by (31) and (32). $\square$

Lemma 3. *Let $u \in \mathbb{P}'$ be quasi-definite and let $\{\Psi_n\}$ be the corresponding sequence defined in (12). If $\{\Psi_n\}$ satisfies the second-order differential equation (25) with coefficients (26)-(28), then the following equation holds:*

$$\hat{A}_n\Psi_n' = \mathcal{M}_n\Psi_n + \mathcal{N}_n\Psi_{n-1}\,, \quad n \geq 1\,,$$

where $\hat{A}_n \in \mathbb{P}$, $\mathcal{M}_n$ is a matrix of order two with polynomial entries, and $\mathcal{N}_n$ is a scalar matrix.

Proof. We write the equation (25) in the form

$$\mathcal{D}_n\varphi_n'' + \mathcal{E}_n\varphi_n' + \mathcal{F}_n\varphi_n = 0_{4\times1} \tag{55}$$

where $\varphi_n = \begin{bmatrix}\Psi_{n+1}\\ \Psi_n\end{bmatrix}$, $n \geq 1$, and $\mathcal{D}_n, \mathcal{E}_n, \mathcal{F}_n$ are block matrices given by

$$\mathcal{D}_n = A^2\begin{bmatrix}\Theta_{n+1}\,I & 0_{2\times2}\\ 0_{2\times2} & \Theta_n\,I\end{bmatrix},\ \mathcal{E}_n = \begin{bmatrix}\tilde{\mathcal{B}}_{n+1} & 0_{2\times2}\\ 0_{2\times2} & \tilde{\mathcal{B}}_n\end{bmatrix},\ \mathcal{F}_n = \begin{bmatrix}\tilde{\mathcal{C}}_{n+1} & 0_{2\times2}\\ 0_{2\times2} & \tilde{\mathcal{C}}_n\end{bmatrix}.$$

Taking $n+1$ in (55) and using the recurrence relations for φ_n (cf. (14)) we obtain

$$\mathcal{D}_{n+1}\mathcal{K}_{n+1}\varphi_n'' + (2\mathcal{D}_{n+1}\mathcal{K}_{n+1}' + \mathcal{E}_{n+1}\mathcal{K}_{n+1})\varphi_n' + (\mathcal{E}_{n+1}\mathcal{K}_{n+1}' + \mathcal{F}_{n+1}\mathcal{K}_{n+1})\varphi_n = 0_{4\times1}\,. \tag{56}$$

To eliminate φ_n'' between (55) and (56) we proceed in two steps: firstly we multiply (55) by $\Theta_{n+2}\mathcal{K}_{n+1}\mathcal{G}_n$, $\mathcal{G}_n = \begin{bmatrix}\Theta_n I & 0\\ 0 & \Theta_{n+1}I\end{bmatrix}$, thus obtaining

$$A^2\Theta_n\Theta_{n+1}\Theta_{n+2}\mathcal{K}_{n+1}\varphi_n'' + \Theta_{n+2}\mathcal{K}_{n+1}\mathcal{G}_n\mathcal{E}_n\varphi_n' + \Theta_{n+2}\mathcal{K}_{n+1}\mathcal{G}_n\mathcal{F}_n\varphi_n = 0_{4\times1}\,, \tag{57}$$

and we multiply (56) by $\Theta_n \mathcal{G}_{n+1}$, thus obtaining

$$\begin{aligned}(58)\quad A^2\Theta_n\Theta_{n+1}\Theta_{n+2}\mathcal{K}_{n+1}\varphi_n'' + \Theta_n\mathcal{G}_{n+1}(2\mathcal{D}_{n+1}\mathcal{K}_{n+1}' + \mathcal{E}_{n+1}\mathcal{K}_{n+1})\varphi_n' \\ + \Theta_n\mathcal{G}_{n+1}(\mathcal{E}_{n+1}\mathcal{K}_{n+1}' + \mathcal{F}_{n+1}\mathcal{K}_{n+1})\varphi_n = 0_{4\times 1}\,.\end{aligned}$$

Then, we subtract (58) to (57), thus obtaining

$$\mathcal{H}_n\varphi_n' = \mathcal{J}_n\varphi_n\,, \tag{59}$$

with

$$\begin{aligned}\mathcal{H}_n &= \Theta_{n+2}\mathcal{K}_{n+1}\mathcal{G}_n\mathcal{E}_n - \Theta_n\mathcal{G}_{n+1}(2\mathcal{D}_{n+1}\mathcal{K}_{n+1}' + \mathcal{E}_{n+1}\mathcal{K}_{n+1})\,,\\ \mathcal{J}_n &= \Theta_n\mathcal{G}_{n+1}(\mathcal{E}_{n+1}\mathcal{K}_{n+1}' + \mathcal{F}_{n+1}\mathcal{K}_{n+1}) - \Theta_{n+2}\mathcal{K}_{n+1}\mathcal{G}_n\mathcal{F}_n\,.\end{aligned}$$

But

$$\mathcal{H}_n^{(2,1)} = \mathcal{H}_n^{(2,2)} = \mathcal{J}_n^{(2,1)} = \mathcal{J}_n^{(2,2)} = 0_{2\times 2}\,,\quad n\geq 1\,,$$

thus Eq. (59) reads

$$\mathcal{H}_n^{(1,1)}\Psi_{n+1}' + \mathcal{H}_n^{(1,2)}\Psi_n' = \mathcal{J}_n^{(1,1)}\Psi_{n+1} + \mathcal{J}_n^{(1,2)}\Psi_n\,. \tag{60}$$

If we write (60) for $n+1$ and if we use the recurrence relation (13) we obtain

$$\mathcal{S}_n\varphi_n' = \mathcal{T}_n\varphi_n \tag{61}$$

with

$$\begin{aligned}\mathcal{S}_n &= \begin{bmatrix}\mathcal{H}_n^{(1,1)} & \mathcal{H}_n^{(1,2)}\\ (x-\beta_{n+2})\mathcal{H}_{n+1}^{(1,1)} + \mathcal{H}_{n+1}^{(1,2)} & -\gamma_{n+2}\mathcal{H}_{n+1}^{(1,1)}\end{bmatrix},\\ \mathcal{T}_n &= \begin{bmatrix}\mathcal{J}_n^{(1,1)} & \mathcal{J}_n^{(1,2)}\\ (x-\beta_{n+2})\mathcal{J}_{n+1}^{(1,1)} + \mathcal{J}_{n+1}^{(1,2)} - \mathcal{H}_{n+1}^{(1,1)} & -\gamma_{n+2}\mathcal{J}_{n+1}^{(1,1)}\end{bmatrix},\end{aligned}$$

where $\det(\mathcal{S}_n)$ is non-zero. The multiplication of (61) by $\mathrm{adj}(\mathcal{S}_n)$ yields

$$\hat{A}_n\varphi_n' = \hat{\mathcal{L}}_n\varphi_n\,,$$

with

$$\hat{A}_n = \det(\mathcal{S}_n),\ \hat{\mathcal{L}}_n = \mathrm{adj}(\mathcal{S}_n)\mathcal{T}_n\,.$$

Thus, the assertion follows. □

Now we study the coefficients of the structure relations obtained in the preceding lemma.

Lemma 4. *Let $u\in\mathbb{P}'$ be quasi-definite and let $\{\Psi_n\}$ be the corresponding sequence defined in (12). Let $\varphi_n = \begin{bmatrix}\Psi_{n+1}\\ \Psi_n\end{bmatrix}$ satisfy*

$$\hat{A}_n\varphi_n' = \hat{\mathcal{L}}_n\varphi_n,\quad n\geq 1\,, \tag{62}$$

where $\hat{A}_n$ are bounded degree polynomials and $\hat{\mathcal{L}}_n$, $n\geq 1$, are block matrices of order two whose entries are bounded degree polynomials. Then, (62) is equivalent to

$$\hat{A}\varphi_n' = \mathcal{L}_n\varphi_n\,,\quad n\geq 1\,, \tag{63}$$

with $\hat{A}\in\mathbb{P}$ and $\mathcal{L}_n$ block matrices of order two whose entries are bounded degree polynomials. Furthermore, there holds

$$\hat{A}\mathcal{K}_{n+1}' = \mathcal{L}_{n+1}\mathcal{K}_{n+1} - \mathcal{K}_{n+1}\mathcal{L}_n\,,\quad n\geq 1\,, \tag{64}$$

where $\mathcal{K}_n$ are the matrices of the recurrence relation (14).

Proof. Taking $n+1$ in (62) and using the recurrence relation for φ_n (cf. (14)) we get

$$\hat{A}_{n+1}\varphi'_n = \mathcal{K}_{n+1}^{-1}\left(\hat{\mathcal{L}}_{n+1}\mathcal{K}_{n+1} - \hat{A}_{n+1}\mathcal{K}'_{n+1}\right)\varphi_n\,. \tag{65}$$

From (62) and (65) we conclude that there exists a polynomial L_n such that

$$\begin{cases} \hat{A}_{n+1} = L_n\hat{A}_n \\ \mathcal{K}_{n+1}^{-1}\left(\hat{\mathcal{L}}_{n+1}\mathcal{K}_{n+1} - \hat{A}_{n+1}\mathcal{K}'_{n+1}\right) = L_n\hat{\mathcal{L}}_n\,, \quad n \geq 1\,, \end{cases}$$

because the first order differential equation for φ_n is unique, up to a multiplicative factor. But from $\hat{A}_{n+1} = L_n\hat{A}_n$ we obtain

$$\hat{A}_{n+1} = (L_n \cdots L_2)\,\hat{A}_1\,,\ \forall n \geq 1\,.$$

Since, for all $n \geq 1$, the degree of $\hat{A}_n$ is bounded by a number independent of n, then the degree of the L_n's must be zero, that is, L_n is constant, for all $n \geq 1$. Hence we obtain (63) with

$$\hat{A} = \hat{A}_1\,,\ \ \mathcal{L}_n = \mathcal{K}_{n+1}^{-1}\left(\hat{\mathcal{L}}_{n+1}\mathcal{K}_{n+1} - \hat{A}_{n+1}\mathcal{K}'_{n+1}\right)/(L_n\cdots L_2)\,.$$

To obtain (64) we take derivatives on $\varphi_{n+1} = \mathcal{K}_{n+1}\varphi_n$ and multiply the resulting equation by $\hat{A}$, to get

$$\hat{A}\,\varphi'_{n+1} = \hat{A}\,\mathcal{K}'_{n+1}\varphi_n + \mathcal{K}_{n+1}\hat{A}\,\varphi'_n\,.$$

Using (63) in the previous equation and the recurrence relation (14) there follows

$$\mathcal{L}_{n+1}\mathcal{K}_{n+1}\varphi_n = \hat{A}\mathcal{K}'_{n+1}\varphi_n + \mathcal{K}_{n+1}\mathcal{L}_n\varphi_n\,,$$

thus (64). □

Corollary 4. *Let $\{\varphi_n\}$ satisfy (63), $\hat{A}\varphi'_n = \mathcal{L}_n\varphi_n\,,\ n \geq 1\,,$ where $\mathcal{L}_n$ are block matrices of order two whose entries are bounded degree polynomials. Then, the following assertions take place:*
(a) $\mathcal{L}_n^{(1,2)}$ is a scalar matrix if, and only if, $\mathcal{L}_n^{(2,1)}$ is scalar.
(b) If $\mathcal{L}_n^{(2,1)}$ is a scalar matrix, then there exist polynomials p_i, $i = 1, \ldots, 3$, such that

$$\mathcal{L}_n^{(1,1)} = \begin{bmatrix} l_{n+1} - p_1 & p_2 \\ p_3 & l_{n+1} + p_1 \end{bmatrix}\,, \quad n \geq 1\,. \tag{66}$$

Proof. Taking into account the definition of $\mathcal{K}_n$, (63) is equivalent to

$$\hat{A}I = (x - \beta_{n+2})(\mathcal{L}_{n+1}^{(1,1)} - \mathcal{L}_n^{(1,1)}) + \mathcal{L}_{n+1}^{(1,2)} + \gamma_{n+2}\mathcal{L}_n^{(2,1)}\,, \tag{67}$$

$$-\gamma_{n+2}\mathcal{L}_{n+1}^{(1,1)} - (x - \beta_{n+2})\mathcal{L}_n^{(1,2)} + \gamma_{n+2}\mathcal{L}_n^{(2,2)} = 0\,, \tag{68}$$

$$(x - \beta_{n+2})\mathcal{L}_{n+1}^{(2,1)} + \mathcal{L}_{n+1}^{(2,2)} - \mathcal{L}_n^{(1,1)} = 0\,, \tag{69}$$

$$-\gamma_{n+2}\mathcal{L}_{n+1}^{(2,1)} - \mathcal{L}_n^{(1,2)} = 0\,. \tag{70}$$

Assertion (a) follows taking into account (70), that is, $\mathcal{L}_n^{(1,2)} = -\gamma_{n+2}\mathcal{L}_{n+1}^{(2,1)}$.

To prove assertion (b) we note that since $\mathcal{L}_n^{(1,2)}$ and $\mathcal{L}_n^{(2,1)}$ are diagonal, from (67) there follows that the entries $(1,2)$ and $(2,1)$ of the matrix $\mathcal{L}_n^{(1,1)}$ are independent of n. Further, from (67) we obtain that $\left[\mathcal{L}_n^{(1,1)}\right]^{(1,1)} - \left[\mathcal{L}_n^{(1,1)}\right]^{(2,2)}$ is independent of n. Hence, (66) follows. □

Lemma 5. *Let $u \in \mathbb{P}'$ be quasi-definite and let $\{q_n\}$ be the corresponding sequence of functions of the second kind. If $\{q_n\}$ satisfies the second-order differential equation (29) with coefficients (30)-(32), then the $\mathcal{Q}_n$'s given in (12) satisfy*

$$\widehat{A}_n \mathcal{Q}_n' = \widehat{\mathcal{L}}_n \mathcal{Q}_n, \quad n \geq 1,$$

with $\widehat{A}_n \in \mathbb{P}$ and $\widehat{\mathcal{L}}_n$ a matrix of order two with analytic entries.

Proof. Analogous to the proof of Lemma 3. □

Lemma 6. *Let $u \in \mathbb{P}'$ be quasi-definite and let $\{\mathcal{Q}_n\}$ be the corresponding sequence given in (12). Let*

$$\widehat{A}_n \mathcal{Q}_n' = \widehat{\mathcal{L}}_n \mathcal{Q}_n, \quad n \geq 1, \tag{71}$$

with $\widehat{A}_n \in \mathbb{P}$ and $\widehat{\mathcal{L}}_n$ a matrix of order two with analytic entries. Then, (71) is equivalent to

$$\widehat{A} \mathcal{Q}_n' = \tilde{\mathcal{L}}_n \mathcal{Q}_n, \quad n \geq 1,$$

with $\widehat{A} \in \mathbb{P}$ and $\tilde{\mathcal{L}}_n$ a matrix of order two with analytic entries. Furthermore, it holds

$$\widehat{A} \mathcal{A}_{n+1}' = \tilde{\mathcal{L}}_{n+1} \mathcal{A}_{n+1} - \mathcal{A}_{n+1} \tilde{\mathcal{L}}_n, \quad n \geq 1,$$

where $\mathcal{A}_n$ are the matrices of the recurrence relation (15).

Proof. Analogous to the proof of Lemma 4. □

Proof of Theorem 2:
Lemma 2 proves $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$. Using the Lemmas 3 and 4 and Corollary 4 we prove $(b) \Rightarrow (a)$. Lemmas 5 and 6 prove $(c) \Rightarrow (a)$.

4. Laguerre-Hahn orthogonal polynomials of class zero

In this section we begin by introducing a matrix type second order operator to Laguerre-Hahn families of orthogonal polynomials of class zero.

Theorem 3. *Let u be a Laguerre-Hahn Stieltjes functional satisfying $\mathcal{D}(Au) = \psi u + B(x^{-1}u^2)$, with $\deg(\psi) = 1$, $\max\{deg(A), \deg(B)\} \leq 2$. Let $\{P_n\}$ be the SMOP related to u and let $\{P_n^{(1)}\}$ be the sequence of associated polynomials of the first kind. It holds that*

$$\mathbb{L}_n(\Psi_n) = 0, \quad \Psi_n = \begin{bmatrix} P_{n+1} \\ P_n^{(1)} \end{bmatrix}, \quad n \geq 0, \tag{72}$$

where $\mathbb{L}_n$ is a matrix operator given by

$$\mathbb{L}_n = A\mathbb{D}^2 + \Psi\mathbb{D} + \Lambda_n \mathbb{I}, \quad \Psi = \begin{bmatrix} \psi & 2B \\ -2D & 2A' - \psi \end{bmatrix}, \; \Lambda_n = \begin{bmatrix} \lambda_{n+1} & B' \\ 0 & \lambda_{n+1}^* \end{bmatrix} \tag{73}$$

where $\mathbb{D}^k$ denotes the derivative operator, $\mathbb{D}^0 = \mathbb{I}$, and

$$\lambda_{n+1} = \lambda_{n+1}^* - A'' + \psi', \lambda_{n+1}^* = 2(n+1)D - n(n+3)\frac{A''}{2} + n\psi', D = \frac{A''}{2} - \psi' - \frac{B''}{2}.$$

Moreover, the three-term recurrence relation coefficients of the MOP sequences $\{P_n\}$ *satisfying (72) are given by*

$$\gamma_n = \frac{(2D - 2(n-1)a_2 + \psi_1)\,\nu_{n-1} + \left(\lambda_n^* - \lambda_{n+1}^*\right)\nu_n}{\lambda_{n-1}^* - \lambda_n^*} + \frac{(2\beta_0 D + 2(n-1)a_1 - \psi_0)\,\alpha_{n-1} - 2\gamma_1 D - 2(n-1)a_0}{\lambda_{n-1}^* - \lambda_n^*}\,, \quad n \geq 2\,, \tag{74}$$

$$\beta_n = \alpha_n - \alpha_{n-1}\,, \quad n \geq 1\,, \tag{75}$$

where, for all $n \geq 1$,

$$\alpha_n = \frac{n\,[-(n+1)a_1 + \psi_0 - 2\beta_0 D]}{-(n-1)(n+2)a_2 + (n-1)\psi_1 - \lambda_{n+1}^* + 2nD}\,, \quad \alpha_0 = 0\,, \tag{76}$$

$$\nu_n = \frac{(n-1)\,[\alpha_n\,(na_1 - \psi_0 + 2\beta_0 D) - na_0 - 2\gamma_1 D]}{(n-2)(n+1)a_2 - (n-2)\psi_1 + \lambda_{n+1}^* - 2(n-1)D}\,. \tag{77}$$

Remark 1. We emphasize the equation enclosed by (72),

$$\mathcal{L}_n^*(P_n^{(1)}) = 2DP_{n+1}'\,, \quad n \geq 0\,, \tag{78}$$

where $\mathcal{L}_n^*$ is the operator defined by

$$\mathcal{L}_n^* = A\mathcal{D}^2 + (2A' - \psi)\mathcal{D} + \lambda_{n+1}^*\mathcal{I}\,.$$

The preceding theorem gives us the formulas for the three-term recurrence relation coefficients of the SMOP $\{P_n\}$ satisfying (78). Note that the full description of the three-term recurrence relation coefficients of Laguerre-Hahn polynomials of class zero was given in [4]. The Eq. (78) has been given in [23] for the classical orthogonal polynomials.

Proof. The Stieltjes function of u satisfies

$$AS' = BS^2 + CS + D\,, \;\; C = \psi - A'\,, \;\; D \text{ constant}\,.$$

Since the class of u is zero, the Θ's involved in the structure relation (16) are constant. If we use the notation $\tau_n = \dfrac{\Theta_{n-1}\Theta_n}{\gamma_n} - l_n^2 + (C/2)^2$, then, taking into account $\tau_n = A\sum_{k=1}^{n} \Theta_{k-1}/\gamma_k + AD + BD$ (cf. (41)), the second-order differential equation (25) can be written as (72) with the operator $\mathbb{L}_n$ given by (73).

To obtain the three-term recurrence relation coefficients of $\{P_n\}$ we start by writing

$$\begin{aligned} P_n^{(1)}(x) &= x^n - \alpha_n x^{n-1} + \nu_n x^{n-2} + \cdots \\ P_{n+1}(x) &= x^{n+1} - (\alpha_n + \beta_0)x^n + (\nu_n + \beta_0\alpha_n - \gamma_1)x^{n-1} + \cdots \end{aligned}$$

with

$$\alpha_n = \sum_{k=1}^{n} \beta_k\,, \;\; \nu_n = \sum_{1\leq i<j\leq n} \beta_i\beta_j - \sum_{k=2}^{n} \gamma_k\,, \quad n \geq 1\,. \tag{79}$$

Equating coefficients of x^{n-1} and x^{n-2} in (78) we get (76) and (77).

To obtain (74) we start by taking derivatives in (7), then multiply the resulting equation by $2D$ and use the equation enclosed by (72)

$$\mathcal{L}_n^*(P_n^{(1)}) = 2DP_{n+1}' \,, \quad \mathcal{L}_n^* = A\mathcal{D}^2 + (2A' - \psi)\mathcal{D} + \lambda_{n+1}^*\mathcal{I} \,, \quad n \geq 0 \,,$$

as well as the recurrence relation, to get

$$2DP_n = 2A\left(P_{n-1}^{(1)}\right)' + (2A'-\psi)P_{n-1}^{(1)} + (\lambda_{n+1}^* - \lambda_n^*)P_n^{(1)} + (\lambda_{n-1}^* - \lambda_n^*)\gamma_n P_{n-2}^{(1)} \,. \tag{80}$$

Equating coefficients of x^{n-2} in (80) we get (74). (75) follows from (79). $\square$

4.1. **Characterizations of classical orthogonal polynomials.** Let us now show a characterization for a sub-family of the Laguerre-Hahn orthogonal polynomials of class zero, namely, the so-called classical families. The results that follow, although not new in the literature, aim to illustrate the preceding theorem.

Hereafter we denote the monic Hermite, Laguerre, Jacobi and Bessel polynomials by $H_n, L_n^\alpha, P_n^{(\alpha,\beta)}$ and B_n^α, respectively. The corresponding three-term recurrence relation coefficients β_n, γ_{n+1}, $n \geq 0$, are given in the table that follows.

	β_n	γ_{n+1}
H_n	0	$\frac{n+1}{2}$
$L_n^{(\alpha)}$	$2n+\alpha+1$	$(n+1)(n+\alpha+1)$
$P_n^{(\alpha,\beta)}$	$\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$	$\frac{4(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)}$
$B_n^{(\alpha)}$	$\frac{-2\alpha}{(n+\alpha)(2n+\alpha+2)}$	$\frac{-4(n+1)(n+\alpha+1)}{(2n+\alpha+1)(2n+\alpha+2)^2(2n+\alpha+3)}$

Table 1

Theorem 4. *Let $u \in \mathbb{P}'$ be regular, let $\{P_n\}$ be the SMOP with respect to u, let $\{P_n^{(1)}\}$ be the sequence of associated polynomials of the first kind, and let $\{q_n\}$ be the sequence of functions of the second kind. The following statements are equivalent:*
(a) u is classical and it satisfies $\mathcal{D}(Au) = \psi u$;
(b) P_n satisfies

$$AP_n'' + \psi P_n' + \lambda_n P_n = 0 \,, \quad n \geq 0 \,; \tag{81}$$

(c) q_n satisfies the second-order differential equation

$$Aq_n'' + (2A' - \psi)q_n' + (\lambda_n + A'' - \psi')q_n = 0 \,, \quad n \geq 0 \,; \tag{82}$$

(d) the derivative P_n' is linked to the polynomial $P_n^{(1)}$ through a relation of the same type as (78),

$$A\left(P_n^{(1)}\right)'' + (2A' - \psi)\left(P_n^{(1)}\right)' + (\lambda_{n+1} + A'' - \psi')P_n^{(1)} = 2DP_{n+1}' \,, \quad n \geq 0 \,, \tag{83}$$

where, for all $n \geq 0$,

$$\lambda_n = -\frac{(n-1)n}{2}A'' - n\psi' \,, \ \ D = \frac{A''}{2} - \psi' \,.$$

Proof. Note that $\mathcal{D}(Au) = \psi u$ is equivalent to the first order differential equation for the corresponding Stieltjes function $AS' = CS + D\,, \quad C = \psi - A'$. Since u is classical, that is, $\deg(A) \le 2, \deg(\psi) = 1$, then the Θ_n's and the l_n's involved in the coefficients of the second-order differential equations (25) and (29) satisfy $\deg(\Theta_n) = 0, \deg(l_n) \le 1$. Thus, (25) yields (81) and (83), and (29) yields (82) for all $n \ge 1$. Notice that (82) for $n = 0$ (with $\lambda_0 = 0$) reads as $AS'' + (A' - C)S' - C'S = 0$, which is the derivative of $AS' = CS + D$.

To prove $(d) \Rightarrow (a)$ we use the equations (74) and (75) (cf. Remark 1) with the values of β_0 and γ_1 given in table 1, thus recovering the expressions for γ_{n+1} and β_n for all $n \ge 1$, thus obtaining the classical families of orthogonal polynomials. $\square$

Remark 2. Eqs. (81)-(83) are known in the literature, see, e.g., [13, 22, 23]. In [22] it is given that the sequence of functions of the second kind, therein denoted by Q_n, constitute another solution (together with P_n) of the hypergeometric-type differential equation $Ay'' + \psi y' + \lambda_n y = 0$. It is well to remark that the notation of [22] for the functions of the second kind, Q_n (cf. chapter II, §11), differs from the notation used in the present paper, q_n. Indeed, one has $Q_n = q_n/w$, being w a solution of the Pearson equation $(Aw)' = \psi w$. Hence, the fact that Q_n is another solution of the hypergeometric-type differential equation $Ay'' + \psi y' + \lambda_n y = 0$ follows from (82) together with w satisfying the Pearson equation.

4.2. **Further illustration of the results: a characterization for Legendre polynomials.** The main purpose of this subsection is to illustrate Theorem 1 for Legendre polynomials.

We denote by $p_n(x) = k_n x^n + \ldots,\ n \ge 0$, the Legendre polynomials. It is well-known that such polynomials are a particular case of Jacobi polynomials, as Jacobi polynomials are orthogonal with respect to the weight function $w(x) = (1-x)^\alpha (1+x)^\beta, x \in [-1, 1]$, and Legendre polynomials are obtained by taking $\alpha = \beta = 0$. Let $p_n^{(1)},\ n \ge 0$, denote the associated polynomials of the first kind related to $\{p_n\}$.

An explicit formula for $\{p_n^{(1)}\}$ is [24, Eq. (8.30)]

$$p_{n-1}^{(1)}(x) = \sum_{j=0}^{n-1} \frac{(2j+1)(1-(-1)^{n+j})}{(n+j+1)(n-j)} p_j(x)\,, \quad n \ge 1\,. \tag{84}$$

In order to proceed with further computations, it is well to recall some facts:
- the three term recurrence relation for $\{p_n\}$ [24, page 146]

$$p_{n+1}(x) = (a_n x + b_n) p_n(x) - c_n p_{n-1}(x), \quad n \ge 1\,, \quad p_0(x) = 1,\ p_1(x) = x\,,$$

with $a_n = (2n+1)/(n+1),\ b_n = 0,\ c_n = n/(n+1)$;
- there holds, for all $n \ge 1$,

$$p_n(x) = k_n x^n + \ldots\,, \quad p_{n-1}^{(1)}(x) = k_n x^{n-1} + \ldots\,, \quad k_{-1} = 0, k_0 = k_1 = 1\,,$$

where $k_{n+1}/k_n = a_n\,, \quad n \ge 0$ (cf. [24, page 135]).

Let us now re-write (84) for the monic case, in our notation, $P_{n-1}^{(1)}$:

$$P_{n-1}^{(1)}(x) = \sum_{j=0}^{n-1} \frac{(2j+1)(1-(-1)^{n+j})}{(n+j+1)(n-j)} \frac{k_j}{k_n} P_j(x)\,, \quad n \ge 1\,, \tag{85}$$

where P_j denotes the monic Legendre polynomial of degree j.

Recall the three term recurrence relation for monic Legendre polynomials,

$$P_{n+1}(x) = xP_n - \gamma_n P_{n-1}(x)\,, \quad \gamma_{n+1} = \frac{(n+1)^2}{(2n+1)(2n+3)}\,, \quad n \geq 0\,. \tag{86}$$

It is well-known that $\{P_n\}$ is related to the Stiletjes function $S(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$ [24, Ch. 8].

Theorem 1 gives us the following.

Corollary 5. *Let $\{P_n\}$ be the seguence of monic Legendre polynomials, let $\{P_n^{(1)}\}$ be the sequence of associated polynomials of the first kind, and let $\{q_n\}$ be the sequence of functions of the second kind. Let S be the Stieltjes function related to $\{P_n\}$. The following statements are equivalent:*
(a) S satisfies

$$AS' = CS + D, \quad A(x) = 1 - x^2,\ C(x) = 0,\ D(x) = 1\,;$$

(b) P_n and $P_n^{(1)}$ satisfy

$$(1-x^2)P'_{n+1} = l_n P_{n+1} + \Theta_n P_n\,,\ (1-x^2)\left(P_n^{(1)}\right)' = P_{n+1} + l_n P_n^{(1)} + \Theta_n P_{n-1}^{(1)}\,, \tag{87}$$

$n \geq 0$, with $l_n(x) = -(n+1)x,\ \Theta_n = (2n+3)\gamma_{n+1}\,;$
(c) q_n satisfies

$$(1-x^2)q'_n = l_{n-1}q_n + \Theta_{n-1}q_{n-1}\,, \quad n \geq 0\,, \tag{88}$$

with $q_{-1} = 1,\ \Theta_{-1} = 1,\ l_{-1} = 0.$

Proof. Note that (88) follows from (87), as $\{q_n\}$ is defined by (9). The first relation in (87) is well-known (see, e.g., [21] and [10, page 94]). Thus, for checking purposes, it remains to check the second relation in (87), taking into account (85).

Let us use

$$P_n^{(1)}(x) = \sum_{j=0}^{n} \lambda_{n,j} P_j(x), \quad \lambda_{n,j} = \frac{(2j+1)(1-(-1)^{n+1+j})}{(n+j+2)(n+1-j)}\frac{k_j}{k_{n+1}}$$

into the second equation of (87). We get

$$(1-x^2)\left(\sum_{j=0}^{n}\lambda_{n,j}P_j(x)\right)' = P_{n+1}(x) + l_n(x)\left(\sum_{k=0}^{n}\lambda_{n,j}P_j(x)\right) + \Theta_n\left(\sum_{j=0}^{n-1}\lambda_{n-1,j}P_j(x)\right)$$

thus, using the first equation in (87), we get

$$\sum_{j=0}^{n}\lambda_{n,j}\underbrace{(1-x^2)P'_j(x)}_{=l_{j-1}(x)P_j(x)+\Theta_{j-1}P_{j-1}(x)} = P_{n+1} + l_n(x)\sum_{j=0}^{n}\lambda_{n,j}P_j + \Theta_n\sum_{j=0}^{n-1}\lambda_{n-1,j}P_j\,.$$

By using $l_n(x) = -(n+1)x,\ n \geq 0$, as well as (86), there follows

$$-\sum_{j=0}^{n} j\lambda_{n,j}P_{j+1}(x) - \sum_{j=0}^{n} j\lambda_{n,j}\gamma_j P_{j-1}(x) + \sum_{j=0}^{n}\lambda_{n,j}\Theta_{j-1}P_{j-1}(x) = P_{n+1}(x)$$
$$-(n+1)\sum_{j=0}^{n}\lambda_{n,j}P_{j+1}(x) - (n+1)\sum_{j=0}^{n}\lambda_{n,j}\gamma_j P_{j-1}(x) + \Theta_n\sum_{j=0}^{n-1}\lambda_{n-1,j}P_j(x)\,.$$

Upon rearranging, one has

$$\lambda_{n,1}(\Theta_0-\gamma_1)P_0(x)+\sum_{j=1}^{n-1}\left(-(j-1)\lambda_{n,j-1}-(j+1)\lambda_{n,j+1}\gamma_{j+1}+\lambda_{n,j+1}\Theta_j\right)P_j(x)$$
$$-(n-1)\lambda_{n,n-1}P_n-n\lambda_{n,n}P_{n+1}=\left(-(n+1)\lambda_{n,1}\gamma_1+\Theta_n\lambda_{n-1,0}\right)P_0(x)$$
$$+\sum_{j=1}^{n-1}\left(-(n+1)\lambda_{n,j-1}-(n+1)\lambda_{n,j+1}\gamma_{j+1}+\Theta_n\lambda_{n-1,j}\right)P_j(x)$$
$$-(n+1)\lambda_{n,n-1}P_n+(1-(n+1)\lambda_{n,n})P_{n+1}\,.$$

As $\{P_n\}$ is a basis, then we obtain the equations for the coefficients of the above linear combination:

$$\lambda_{n,1}(\Theta_0-\gamma_1)=-(n+1)\lambda_{n,1}\gamma_1+\Theta_n\lambda_{n-1,0}\,, \tag{89}$$
$$-(j-1)\lambda_{n,j-1}-(j+1)\lambda_{n,j+1}\gamma_{j+1}+\lambda_{n,j+1}\Theta_j \tag{90}$$
$$=-(n+1)\lambda_{n,j-1}-(n+1)\lambda_{n,j+1}\gamma_{j+1}+\Theta_n\lambda_{n-1,j}\,,\ j=1,\ldots,n-1\,, \tag{91}$$
$$-(n-1)\lambda_{n,n-1}=-(n+1)\lambda_{n,n-1}\,, \tag{92}$$
$$-n\lambda_{n,n}=1-(n+1)\lambda_{n,n}\,. \tag{93}$$

Taking into account the above definitions of $\lambda_n,\Theta_n,\gamma_n$, the above relations (89)-(93) are true. Hence, we conclude that the second relation in (87) holds for $\{P_n^{(1)}\}$ defined by (85), as required. $\square$

Remark 3. The sequence $\{P_n^{(1)}\}$ consitutes an example of a Laguerre-Hahn family. Indeed, taking into account (8) and (87), there follows that $\{P_n^{(1)}\}$ is related to the Stieltjes function $S^{(1)}$ that satisfies the Riccati equation $A\left(S^{(1)}\right)'=B\left(S^{(1)}\right)^2+CS^{(1)}+D$ with $A(x)=1-x^2,\ B(x)=1/3,\ C(x)=-2x,\ D(x)=3$.

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